

The Sphaleron Rate: Where We Stand

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I review what we know about the “sphaleron rate”, which is the efficiency of baryon number violation at high temperatures $T \sim 100$ GeV in the Standard Model. The leading behavior at weak coupling in the symmetric phase is known accurately; $\Gamma = (10.7 \pm .7)(g^2 T^2 / m_D^2) \log(m_D / g^2 T) \alpha_w^5 T^4$. At realistic values of the coupling our accuracy is worse. We also now have the tools to determine the rate nonperturbatively in the broken electroweak phase; the sphaleron rate there is slower than perturbative estimates.

1 Introduction

The Universe is filled with matter, and virtually no antimatter. This “unusual” situation can only be explained without appealing to the initial conditions of the Universe if, at some early epoch in the history of the universe, baryon number was not a conserved quantity.

As a matter of fact, baryon number is *not* a conserved quantity in the standard model¹. Furthermore, while its violation under ordinary conditions is pitifully inefficient, that ceases to be true at very high temperatures, of order the weak scale, where electroweak symmetry is restored. These facts are the backdrop for the subject of electroweak baryogenesis, which attempts to use them to explain why the baryon number density of the current universe is what it is.

In this talk I will not discuss baryogenesis. Rather my emphasis is on strengthening its foundations by investigating more accurately exactly how efficiently baryon number is violated under hot conditions. Besides the obvious application to the study of baryogenesis, this is also a useful thing to do because it forces us to develop tools for dealing with the infrared physics of hot Yang-Mills theory. Also it is the only part of a baryogenesis calculation which is generic to all extensions of the standard model, because it only depends on the gauge group, and only very weakly on the Higgs sector. In fact, for much of the talk I will neglect the Higgs fields altogether and just study Yang-Mills theory.

1.1 What we want to know

Before going any further I should fix notation and explain what I will measure. The anomaly equation relates baryon number to the Chern-Simons number of

the SU(2) weak fields, through

$$\frac{1}{3}N_{\text{B}}(t) = N_{\text{CS}} \equiv \frac{1}{8\pi^2} \int^t dt' \int d^3x E_i^a B_i^a(x, t'), \quad (1)$$

where E and B are the SU(2) electric and magnetic fields, and I normalize so the gauge field A has units of inverse length and g^2 appears in the denominator in the Lagrangian. The constant of integration from the indefinite time integral is fixed by requiring N_{CS} to be an integer for a vacuum configuration. N_{CS} is called the Chern-Simons number. Because magnetic fields are always transverse (Gauss' Law for magnetism), the evolution of Chern-Simons number depends on the physics of the transverse sector.

Having defined Chern-Simons number I can define its diffusion constant,

$$\Gamma \equiv \lim_{V \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{\langle (N_{\text{CS}}(t) - N_{\text{CS}}(0))^2 \rangle}{Vt}, \quad (2)$$

where the angular brackets $\langle \rangle$ mean an average is taken over the thermal density matrix. Γ is often referred to as the “sphaleron rate.” The reason we care about it is that there is a fluctuation dissipation relation between it and the relaxation rate for a chemical potential for baryon number. I will not discuss this in detail, see instead ^{2,3,4}. I also comment that for N_{CS} to diffuse requires nonperturbative physics, and nonperturbative physics is only unsuppressed, at high temperatures and weak coupling, on length scales $\geq (1/g^2T)^{1/2}$.

It is interesting to know Γ in two regimes. The first is the electroweak symmetric phase. Almost all baryogenesis mechanisms will give a final baryon number directly proportional to its value here. For this reason we would like to know it with some accuracy here, which makes the calculation tricky. The other regime where we want to know Γ is in the broken electroweak phase, *immediately* after the electroweak phase transition, which is to say, right after the baryons were allegedly produced. In this case what we want to know is, are the baryons safe, or will they subsequently be destroyed? Here the strength of the phase transition is important; the real question is, how strong must the phase transition be to prevent the baryons from getting destroyed? To answer this question with descent resolution we only need to know Γ to within ± 1 in the exponent; however Γ is exponentially small and perturbation theory is not yet reliable, which will make this calculation tricky as well.

1.2 Approximations I will need

Determining Γ requires determining unequal Minkowski time correlators at finite temperature in a quantum field theory. Furthermore, if the answer is to

be nontrivial the field theory must be showing nonperturbative physics. No one knows how to do this directly. Therefore I will obviously need to make some approximations.

What saves us is that the $SU(2)$ sector is weakly coupled. This allows two key approximations which make the problem tractable. First, the infrared behavior of the theory is classical up to parametrically suppressed corrections. Second, the ultraviolet behavior of the theory is perturbative. Here, infrared means $k \ll \pi T$, while ultraviolet means $k \gg \alpha_w T$. When the coupling really is weak, there is an overlap between these two regimes, and every degree of freedom can be treated with one approximation or the other. Then, one can integrate out those degrees of freedom which are perturbative, and treat the remaining, classical theory nonperturbatively on the lattice.

In what follows I will first discuss what we learn by treating perturbatively and integrating out everything we can. Then I will step back and only integrate out the highest k modes analytically, leaving a larger and more inclusive theory for numerical work. Finally I discuss what we can do in the broken phase. The complete details for these three approaches can be found in ⁶, ⁷, and ⁸ respectively.

2 Leading log

Dietrich Bödeker has shown that it is possible to integrate out all degrees of freedom with momentum scale $k \geq g^2 T \log(1/g)$, and that doing so produces an effective theory for the remaining $k \sim g^2 T$ degrees of freedom which is classical Yang-Mills theory under Langevin dynamics⁹. The physical origin of this effective theory has been discussed by Arnold's group¹⁰. Integrating out the modes with $k \geq gT$ gives the well known hard thermal loop effective theory¹¹. The behavior of the infrared modes in this theory is overdamped¹², which just follows from Lenz's law and the fact that the plasma is highly conducting. The conductivity is k dependent on scales shorter than some mean collision length, which in an abelian theory is the large angle scattering length. However in a nonabelian theory a particle's charge is changed by scattering. The mean length for a particle to travel before its charge is randomized is

$$l_{\text{scatt}}^{-1} = \frac{g^2 T}{2\pi} \left[\log \frac{m_D}{g^2 T} + O(1) \right], \quad (3)$$

so on scales longer than this the strength of damping is k independent. Therefore, in the approximation that the scale $1/g^2 T$ is well separated from the scale $1/(g^2 T \log(1/g))$, the infrared fields obey Langevin dynamics on long time scales.

lattice spacing a	Volume	Langevin time	$\kappa' \pm$ statistical error
$2/3g^2T$	$(8/g^2T)^3$	$290000a^2$	10.44 ± 0.23
$2/3g^2T$	$(16/g^2T)^3$	$49500a^2$	10.30 ± 0.21
$2/5g^2T$	$(16/g^2T)^3$	$21000a^2$	10.70 ± 0.67
$2/7g^2T$	$(16/g^2T)^3$	$42000a^2$	10.25 ± 0.79

Table 1: Results for κ' at three lattice spacings and two lattice volumes. The results show excellent spacing and volume independence.

Langevin dynamics have two nice features. First, the Langevin dynamics of 3-D Yang-Mills theory are free of UV problems, and a zero lattice spacing limit exists. Second, Langevin dynamics are very easy to put on the lattice. Therefore the emphasis should be on controlling systematics, such as

1. the thermodynamic match between lattice and continuum,
2. the match between lattice and continuum Langevin time scales,
3. topological definition of N_{CS} , and
4. the large volume and long time limits.

All of these systematics can be controlled. The first is discussed in¹³, the second in⁶, and the third in⁸. A volume $8/g^2T$ on a side is large enough to achieve the large volume limit, so I use a volume $16/g^2T$ on a side for overkill. The failure to achieve the large time limit is reflected in the statistical error bars.

The results, which show beautiful lattice spacing independence, are presented in Table 1, which gives the coefficient κ' for the equation

$$\Gamma = \kappa' \left(\log \frac{m_D}{g^2T} + O(1) \right) \left(\frac{g^2T^2}{m_D^2} \right) \alpha_w^5 T^4. \quad (4)$$

These results settle the question, “What is the sphaleron rate in the symmetric phase, in the extreme weak coupling limit?”

3 Beyond the leading log

In the last section I determined the coefficient of the leading log behavior with very good precision. The problem is the $(+O(1))$ appearing in Eq. (4). How well does the expansion in $\log(1/g)$ converge?

The answer is, probably very poorly. To see this, compare the free path for color randomization, $l_{\max} = 2\pi/g^2T \log(1/g)$, to the size of a box for which

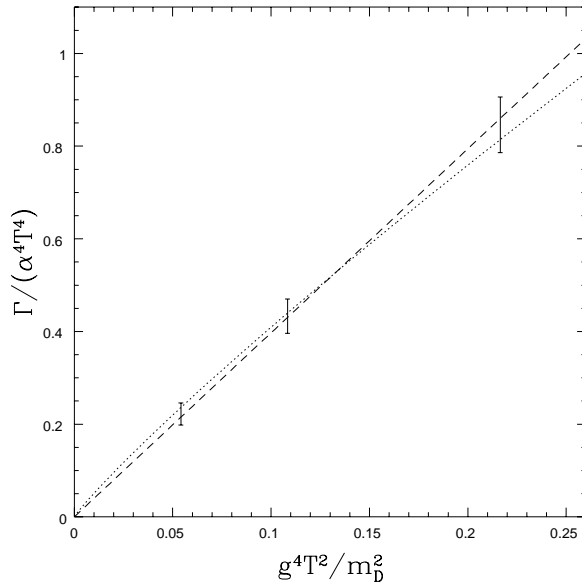


Figure 1: Sphaleron rate plotted against inverse HTL strength, for the lattice theory plus “particles” used to generate the hard thermal loops. The dashed fit is a straight line, while the dotted fit incorporates the known logarithmic dependence on m_D found in the previous section.

the large volume limit has already been reached, $8/g^2T$. There is not a large separation between these scales. In fact, it is not clear whether l_{\max} is smaller than the scale characterizing nonperturbative physics, which must after all be well shorter than the dimension of a box which shows large volume behavior. The problem is that Bödeker’s approach requires integrating out modes for which the perturbative treatment may not be very reliable. To test this, and to try to determine the sphaleron rate beyond leading log, we need to integrate out less, and make the numerical model include the gT as well as g^2T scales.

An effective action for the theory with the $k \sim T$ modes integrated out is known¹⁴, and goes by the name of the hard thermal loop (HTL) effective action. It is nonlocal, which is not surprising, since its construction involves integrating out propagating degrees of freedom in a Minkowski theory. Unfortunately nonlocality is very problematic for a numerical implementation.

A solution to this problem was proposed a few years ago by Hu and Müller¹⁵. The idea is that, rather than add the HTL action itself, one adds a set of local degrees of freedom which, if integrated out, would also yield the HTL ef-

fective action. Since the HTL action represents the propagation of a set of high momentum, charged particles, what we add are a bunch of high momentum, charged classical particles. For the idea to work it is necessary to add particles in such a way that the numerical model retains an exact gauge invariance, and possesses a conserved energy and phase space measure so that thermodynamic averages are well defined. We present an implementation which satisfies these conditions in ⁷, and refer the reader there for the (quite complicated) details.

With a model which reproduces the HTL resummed IR physics in hand, it is possible to test Arnold, Son, and Yaffe’s claim that the IR dynamics are overdamped, directly. The results are shown in Figure 1, which shows that Γ does vanish linearly as m_D^2 is increased. The data are not good enough to show unambiguously whether or not Bödeker’s log is present. Fitting the data assuming it is, the $(+O(1))$ in Eq. (4) turns out to be about 3.6, which indicates that the expansion in $\log(m_D/g^2T)$ is not a very good one. For the physical value of $m_D^2 = (11/6)g^2T^2$ and $g^2 \sim 0.4$, the sphaleron rate is $(20 - 25)\alpha^5T^4$, with systematics dominated errors of the order of 30%. The main remaining problems to be addressed involve lattice spacing effects and the interactions of the added “particle” degrees of freedom with the most UV lattice modes.

4 The broken electroweak phase

In the previous two sections the approach has been to find a numerical system which has the same physics as thermal Yang-Mills theory, and then to evolve it and measure directly the correlator, Eq. (2), which tells how efficiently baryon number is violated. However, this approach fails completely in the broken electroweak phase, because the rate of topological transitions is so small that no reasonable amount of numerical evolution would see any transitions at all. Another alternative, perturbation theory, is not very reliable close to the electroweak phase transition. We know for instance that the one loop and two loop effective potentials give quite different answers for the strength of the transition, and no one knows how to compute the sphaleron rate beyond the one loop level. Some other technique, nonperturbative but not strictly real time, is needed.

The reason for the suppression of the sphaleron rate in the broken phase is shown, in cartoon form, in Figure 2. There is a free energy barrier between minima, meaning that almost none of the weight of the thermal ensemble lies in states intermediate between vacua. The figure also suggests how I will determine the sphaleron rate in the broken phase. I should define N_{CS} (or some appropriate observable) on the lattice, and measure how the free energy

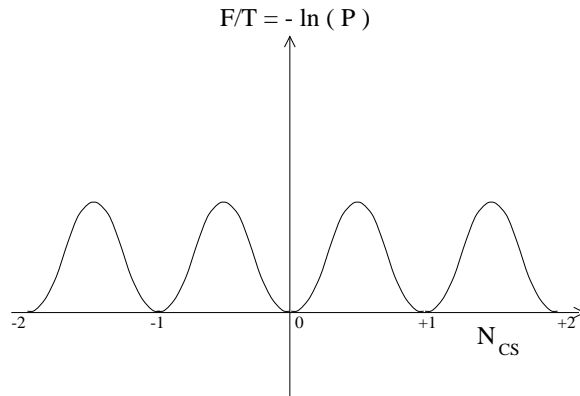


Figure 2: “Cartoon” of the free energy dependence on N_{CS} .

depends on it. This is not enough to give the real time rate, but with some more work one can turn the height of the barrier into the real time rate.

4.1 Defining N_{CS}

I begin by defining Chern-Simons number on the lattice. The obvious approach is to use the same definition as in the continuum, Eq. (1). Note that the integral over “time” in that equation could really be an integral along any path through the space of configurations, not just one generated by Hamilton’s equations. In particular one can fix the constant of integration by having the path begin or end at a vacuum configuration.

There is a problem on the lattice, which is that no lattice implementation of $E_i^a B_i^a$ is exactly a total derivative. Therefore N_{CS} defined through Eq. (1) and implemented on the lattice would depend on the path chosen. The resolution is to choose a unique and particularly sensible path, the gradient flow (cooling) path, that is, the path through configuration space along which the energy falls most rapidly. In continuum notation the path (parameterized by a cooling time τ) is given by

$$\frac{dA(x, \tau)}{d\tau} = -\frac{\partial H}{\partial A(x)}, \quad (5)$$

where H is the Hamiltonian and all indices have been suppressed. Besides being unique, this path also has the benefit that it moves quickly towards configurations where the gauge fields are smooth. This minimizes the impact of

lattice artifacts, which were for instance responsible for $E_i^a B_i^a$ not being a total derivative. Further, the path automatically goes to a vacuum configuration.

This definition of N_{CS} has two problems, both easily resolved. First, N_{CS} is UV poorly behaved. For instance, its mean squared value diverges as V/a , with V the physical volume and a the lattice spacing (or other regulator). This is resolved by using not N_{CS} but the Chern-Simons number of a configuration after an initial length τ_0 of gradient flow. Our measurable is then dependent on an unphysical parameter τ_0 , but it is UV finite, and physical measurables such as Γ will be τ_0 independent in the end.

The second problem is that performing gradient flow down to the vacuum is intensely numerically expensive; yet we will need to do so thousands of times to determine the free energy distribution. This problem is solved by blocking. Gradient flow destroys information, and in particular it destroys almost all the UV information; so nothing is lost by blocking after some modest amount of gradient flow. The numerical savings are immense, and (if we use an $O(a^2)$ improved lattice Hamiltonian and implementation of $E_i^a B_i^a$) almost no accuracy is lost.

Using this definition of N_{CS} , it is possible to determine the free energy (probability distribution) as a function of N_{CS} by standard multicanonical Monte-Carlo techniques. A sample result is shown in Figure 3.

4.2 Turning probabilities into rates

One cannot read off the sphaleron rate from Figure 3 alone; in fact the height of the barrier in the figure depends on an arbitrary parameter τ_0 . To get Γ from the figure we need to know

$$\langle \dot{N} \rangle \equiv \left\langle \left| \frac{dN_{\text{CS}}}{dt} \right|_{N_{\text{CS}}=0.5} \right\rangle, \quad (6)$$

the mean rate at which N_{CS} is changing during a crossing of the barrier. Multiplying the probability density at the top of the barrier by $\langle \dot{N} \rangle$ turns the probability density into a probability flux per unit time.

It is straightforward to measure $\langle \dot{N} \rangle$ numerically. First, we use multicanonical means to get a sample of configurations with $N_{\text{CS}} \simeq 0.5$. Then, for each we draw momenta randomly from the thermal ensemble and perform a very short period of Hamiltonian evolution, measuring N_{CS} before and after. Then $|dN_{\text{CS}}/dt|$ is approximated by $|N_{\text{CS}}(0) - N_{\text{CS}}(\delta t)|/(\delta t)$, and $\langle \dot{N} \rangle$ is the average of this over the sample. Also, one must divide by the volume used in the lattice simulation, to convert the rate of topological transitions to the rate per unit volume.

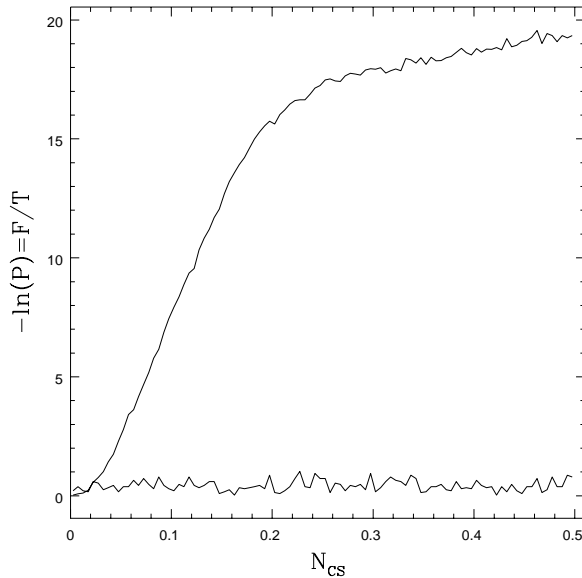


Figure 3: Free energy as a function of N_{CS} , at the critical temperature in a $(16/g^2T)^3$ cubic box and using $\tau_0 = 3.6/(g^2T)^2$, when the ratio of the Higgs self-coupling to the gauge coupling was $\lambda/g^2 = 0.039$. The upper curve is the broken phase value and the lower curve is the symmetric phase value.

However, Γ does not equal the probability flux per unit time over the barrier; it is how often one goes from being in one topological vacuum to being in another. It is possible to cross the barrier several times on the way from one minimum to its neighbor, or to cross an even number of times and return to the starting vacuum. This leads to a correction called the “dynamical prefactor,” which is the ratio of true topological vacuum changes to crossings of the top of the barrier. To compute it, we use multicanonical means to get a sample of $N_{CS} = 0.5$ configurations. Then each is evolved under Hamiltonian dynamics, both forward and backwards in time, until it settles in a topological vacuum. The dynamical prefactor is

$$\text{Prefactor} = \sum_{\text{sample}} \frac{1}{\# \text{ crossings}} (\Delta N_{CS})^2, \quad (7)$$

where ΔN_{CS} is the difference in N_{CS} between the starting and ending vacua. It is ± 1 if there were an odd number of $N_{CS} = 0.5$ crossings and 0 if there were an even number; we never observe prompt crossings from one topological

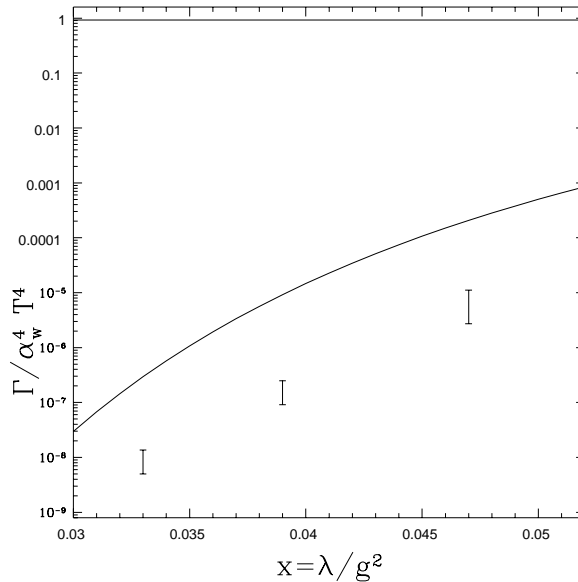


Figure 4: Chern-Simons number diffusion constant Γ immediately after the electroweak phase transition as a function of λ/g^2 , which at tree level equals $m_H^2/8m_W^2$. The solid line is a perturbative estimate, the line at the top of the figure is the symmetric phase rate.

minimum to another which is not its immediate neighbor.

The hard thermal loops appear in the dynamical prefactor, which is parametrically of order $(g^4 T^2/m_D^2) \log(m_D/g^2 T)$. However, using the techniques of the last section to include the HTL effects in the calculation of the prefactor shows that, for realistic values of m_D^2 , the importance of HTL's is weak. This is expected, or at least we should expect that the dependence is weaker than in the symmetric phase, because broken phase baryon number violation should be mediated by a spatially smaller configuration, involving higher frequency modes which are less overdamped.

The final result for Γ in the broken electroweak phase, at the electroweak phase transition temperature and for a range of scalar self-couplings, is plotted in Figure 4, which also compares it to a perturbative result based on a two loop potential and the zero mode calculations from Carson and McLerran¹⁶. The actual rate is substantially but not drastically slower than the perturbative estimate. For comparison, the value needed to avoid baryon number washout after the transition, in the standard cosmology, is $\Gamma \sim 10^{-7} \alpha_w^4 T^4$.

5 Conclusion

Tools now exist to calculate the baryon number violation rate in both the symmetric and broken electroweak phases. In the symmetric phase the rate behaves parametrically as α_w^5 , with a logarithmic correction found by Bödeker which is numerically small. The rate at a realistic m_D^2 is $20 - 25\alpha_w^5 T^4$, with systematic errors, estimated to be of order 30%, dominating statistical errors. In the broken phase the rate is smaller than a perturbative estimate, but still too large to save baryogenesis in the minimal standard model.

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